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Equations of motion for $N = 8$ supergravity from its geometrical description in central charge superspace

Annamária Kiss

Department of Physics, University of Maryland
College Park, MD 20742, USA
E-mail: akiss@physics.umd.edu

Emmanuel Loyer

Laboratoire de Physique, ENS Lyon, 46 Allée d'Italie
69364 Lyon, France
E-mail: eloyer@ens-lyon.fr

Abstract: We consider the geometrical description of $N = 8$ supergravity in central charge superspace and present in detail the deduction from this geometrical formulation of both the supergravity transformations and the equations of motion for the component fields. We compare these results with those already known and obtained in formulations on the component level.

Keywords: Extended Supersymmetry, Supergravity Models, Superspaces.

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Contents

1. Introduction

The steps for the superspace description of all supergravities are the following [1, 2]. First, impose a set of constraints on superspace torsions, curvatures and field strength, second, solve the Bianchi identities on these quantities, third, use the Wess-Zumino gauge to obtain component results. It is important to emphasize that one finds as a consequence of these steps the expression of supercovariant field strengths in terms of component fields as well as the supergravity transformations. Moreover, in the case of on-shell theories the equations of motion are completely determined by these steps without even supposing the existence of a Lagrangien.

For the $N = 8$ supergravity of Cremmer and Julia [3]–[6], where all the 70 helicity 0 states are scalars, there exist two approaches to its superspace description. One of them is performed in ordinary superspace [7, 8] while the other in central charge superspace [8, 9, 10]. The two approaches are equivalent. In both case the scalars are identified in the geometry through their field strength and the description in central charge superspace offers the possiblity to see them as components of the supervielbein in the extra bosonic sector.

The aim of the present article is to complete the existing studies of the $N = 8$ supergravity in superspace by a detailed presentation of the deduction of equations of motion and supergravity transformations from its geometric formulation.

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We use an approach described in detail in the preprint [11]. As usual, the graviton, the gravitini and the graviphotons are identified[1] in the supervielbein $E^{\mathcal{A}} = (E^a, E_{\mathrm{A}}^{\alpha}, E^{\mathrm{A}}_{\dot\alpha}, E^{\mathbf{u}})$ with $\mathbf{u}$ the central charge index running from 1 to 56,

$$E^a\| = \mathrm{d}x^m e_m{}^a\,, \qquad E_{\mathrm{A}}^{\alpha}\| = \frac{1}{2}\,\mathrm{d}x^m \psi_m{}_{\mathrm{A}}^{\alpha}\,, \qquad E^{\mathrm{A}}_{\dot\alpha}\| = \frac{1}{2}\,\mathrm{d}x^m \bar\psi_m{}^{\mathrm{A}}_{\dot\alpha}\,, \qquad E^{\mathbf{u}}\| = \mathrm{d}x^m v_m{}^{\mathbf{u}}\,, \tag{1.1}$$

the helicity 1/2 fields are identified in 1/2 dimensional torsion components,

$$T_{\gamma\ \beta\ \dot\alpha}^{\mathrm{CBA}}| = \epsilon_{\beta\gamma}\lambda^{[\mathrm{CBA}]}{}_{\dot\alpha}\,, \qquad T^{\dot\gamma\dot\beta\alpha}_{\mathrm{CBA}}| = \epsilon^{\dot\beta\dot\gamma}\lambda_{[\mathrm{CBA}]}{}^{\alpha}\,, \tag{1.2}$$

but in contrast with the approaches already mentioned, the scalars are identified directly in 0 canonical dimension torsion components

$$T_{\gamma\beta}^{\mathrm{CB}\,\mathbf{u}}| \ = \ \epsilon_{\gamma\beta}\mathcal{T}^{[\mathrm{CB}]\mathbf{u}}\,, \qquad T^{\dot\gamma\dot\beta\,\mathbf{u}}_{\mathrm{CB}}| = \epsilon^{\dot\gamma\dot\beta}\mathcal{T}_{[\mathrm{CB}]}{}^{\mathbf{u}}\,. \tag{1.3}$$

This approach is a generalization of the construction applied to $N = 2$ [12] and $N = 4$ supergravity with an antisymmetric tensor [13], and has the property that only 0 and 1/2 dimensional constraints are sufficient to identify the components in the geometry as well as to deduce the essential properties of the multiplet: the $E_{7(+7)}/\,\mathrm{SU}(8)$ coset space structure of the scalars and twisted self-duality of the graviphotons. Even if this point of view was fruitful in the research of a minimal set of constraints sufficient to identify the multiplet in the geometry, it turned out that there is a correspondence between the results of this approach and those of the superspace description presented in detail by Howe [10]. The correspondence is based on a redefinition of the frame consisting in a rotation by the scalars in the central charge sector. Therefore here we just include the solution of the Bianchi identities corresponding to this approach in the appendix and concentrate rather on the derivation of the equations of motion.

The structure of this paper is the following. We begin in section 2 by deducing the expression of the supercovariant field strengths in terms of component fields. These are needed both in section 3 where we deduce the supergravity transformations of the components and in section 4 where the equations of motion are calculated. All along the article the component results obtained from the geometric description are compared with those of the component level formulations.

2. Supercovariant → component toolkit

The basic superfields appearing in torsion components (see the appendix) are all supercovariant quantities and their lowest superfield components inherit this property. Let us sum up in this section the component expressions of the supercovariant field strengths, needed to write down supergravity transformations of the component fields. General formulas used to determinate these expressions are easily written using the notation $E^{\mathcal{A}}\| = e^{\mathcal{A}} = \mathrm{d}x^m e_m{}^{\mathcal{A}}$.

[1]The double bar projects at the same time on the vector coefficient of the differential form and on the lowest superfield component.

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Recall that the graviton, gravitini and graviphotons are identified in the supervielbein. Thus, their field strength can be found in the expression of the corresponding supercovariant field strengths counterparts using

$$T^{\mathcal{A}}\| = \frac{1}{2}\mathrm{d}x^m \mathrm{d}x^n \left(\mathcal{D}_n e_m{}^{\mathcal{A}} - \mathcal{D}_m e_n{}^{\mathcal{A}}\right) = \frac{1}{2} e^{\mathcal{B}} e^{\mathcal{C}} T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}}| \,. \tag{2.1}$$

For $\mathcal{A} = a$ one finds the relation

$$\mathcal{D}_n e_m{}^a - \mathcal{D}_m e_n{}^a = i\psi_{[n\mathrm{A}}\sigma^a \bar{\psi}_{m]}{}^{\mathrm{A}} \,, \tag{2.2}$$

which determinates the Lorentz connection $\Phi_{mkl} = e_k{}^b e_l{}^a \Phi_{mba}$ in terms of the vierbein, its derivatives and gravitini fields[2]

$$\begin{aligned} \Phi_{mkl} \;=\; & \frac{1}{2}\left(e_m{}^a \partial_k e_{la} - e_l{}^a \partial_m e_{ka} - e_k{}^a \partial_l e_{ma}\right) - (k \leftrightarrow l) - \\ & - \frac{\mathrm{i}}{4}\left(\psi_{m\mathrm{A}}\sigma_k \bar{\psi}_l{}^{\mathrm{A}} - \psi_{l\mathrm{A}}\sigma_m \bar{\psi}_k{}^{\mathrm{A}} + \psi_{k\mathrm{A}}\sigma_l \bar{\psi}_m{}^{\mathrm{A}}\right) - (k \leftrightarrow l) \,. \end{aligned} \tag{2.3}$$

For $\mathcal{A} = \mathbf{u}$, the central charge indices, we obtain the covariant field strength of the graviphotons

$$\begin{aligned} F_{ba}{}^{\mathbf{u}}| \;=\; & e_b{}^n e_a{}^m \mathcal{F}_{nm}{}^{\mathbf{u}} + e_b{}^n e_a{}^m \left[\frac{1}{4}\left(\bar{\psi}_n{}^{\mathrm{C}} \bar{\psi}_m{}^{\mathrm{B}}\right) + \frac{\mathrm{i}}{8}\left(\psi_{[n\mathrm{A}}\sigma_{m]}\lambda^{[\mathrm{CBA}]}\right)\right] \mathcal{T}_{[\mathrm{CB}]}{}^{\mathbf{u}} + \\ & + e_b{}^n e_a{}^m \left[\frac{1}{4}(\psi_{n\mathrm{C}}\psi_{m\mathrm{B}}) + \frac{\mathrm{i}}{8}\left(\bar{\psi}_{[n}{}^{\mathrm{A}}\bar{\sigma}_{m]}\lambda_{[\mathrm{CBA}]}\right)\right] \mathcal{T}^{[\mathrm{CB}]\mathbf{u}}, \end{aligned} \tag{2.4}$$

with $\mathcal{F}_{nm}{}^{\mathbf{u}}$ the field strength of the graviphotons $\mathcal{F}_{nm}{}^{\mathbf{u}} = \partial_n v_m{}^{\mathbf{u}} - \partial_m v_n{}^{\mathbf{u}}$. This can also be written in the SU(8) basis:

$$F_{ba\,[\mathrm{CB}]}| \;=\; e_b{}^n e_a{}^m \mathcal{F}_{nm}{}^{\mathbf{u}} \mathcal{S}_{\mathbf{u}[\mathrm{CB}]} + e_{[b}{}^n e_{a]}{}^m \left[\frac{1}{4}(\psi_{n\mathrm{C}}\psi_{m\mathrm{B}}) + \frac{\mathrm{i}}{8}\left(\bar{\psi}_n{}^{\mathrm{A}}\bar{\sigma}_m \lambda_{[\mathrm{CBA}]}\right)\right], \tag{2.5}$$

$$F_{ba}{}^{[\mathrm{CB}]}| \;=\; e_b{}^n e_a{}^m \mathcal{F}_{nm}{}^{\mathbf{u}} \mathcal{S}_{\mathbf{u}}{}^{[\mathrm{CB}]} + e_{[b}{}^n e_{a]}{}^m \left[\frac{1}{4}(\bar{\psi}_n{}^{\mathrm{C}}\bar{\psi}_m{}^{\mathrm{B}}) + \frac{\mathrm{i}}{8}\left(\psi_{n\mathrm{A}}\sigma_m \lambda^{[\mathrm{CBA}]}\right)\right]. \tag{2.6}$$

However, the field strength of the graviphotons satisfy self-duality relations, which reduce their degrees of freedom to the half. In particular, the dynamic part of this field strength corresponds to the self-dual part $F^{+}_{ba\,[\mathrm{BA}]}$ of the component $F_{ba\,[\mathrm{BA}]}$ and the anti-self-dual part $F^{-}_{ba}{}^{[\mathrm{BA}]}$ of the component $F_{ba}{}^{[\mathrm{BA}]}$, while the remaining parts (the anti-self-dual part of the component $F_{ba\,[\mathrm{BA}]}$ and the self-dual part of the component $F_{ba}{}^{[\mathrm{BA}]}$) are given as quadratic terms in the spinor superfields. Therefore, we are rather interested in the expression of lowest superfield components of the objects $F^{+}_{ba\,[\mathrm{BA}]}$ and $F^{-}_{ba}{}^{[\mathrm{BA}]}$:

$$F^{+}_{ba\,[\mathrm{CB}]}| \;=\; e_b{}^n e_a{}^m \mathcal{F}^{+}_{nm}{}^{\mathbf{u}} \mathcal{S}_{\mathbf{u}[\mathrm{CB}]} - \frac{1}{8}\mathrm{tr}(\sigma_{ba}\sigma^{nm}) \left[(\psi_{n\mathrm{C}}\psi_{m\mathrm{B}}) + \frac{\mathrm{i}}{2}\left(\bar{\psi}_n{}^{\mathrm{A}}\bar{\sigma}_m \lambda_{[\mathrm{CBA}]}\right)\right], \tag{2.7}$$

$$F^{-}_{ba}{}^{[\mathrm{CB}]}| \;=\; e_b{}^n e_a{}^m \mathcal{F}^{-}_{nm}{}^{\mathbf{u}} \mathcal{S}_{\mathbf{u}}{}^{[\mathrm{CB}]} - \frac{1}{8}\mathrm{tr}(\bar{\sigma}_{ba}\bar{\sigma}^{nm}) \left[\left(\bar{\psi}_n{}^{\mathrm{C}}\bar{\psi}_m{}^{\mathrm{B}}\right) + \frac{\mathrm{i}}{2}\left(\psi_{n\mathrm{A}}\sigma_m \lambda^{[\mathrm{CBA}]}\right)\right]. \tag{2.8}$$

[2]One may observe that in [6], page 334, the Lorentz connection depends also on the spinor fields. However, this difference is just a matter of redefinition, it corresponds in our geometrical description to the replacement of the conventional constraint $T_{cb}{}^a = 0$ by $T_{cb}{}^a = \lambda\varepsilon_{dcb}{}^a (T_{[\mathrm{CBA}]}\sigma^d T^{[\mathrm{CBA}]})$.

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These are the objects, which correspond to the supercovariant field strength of the graviphotons $\hat{F}^{+}_{nm\,[\mathrm{CB}]}$ and $\hat{F}^{-}_{nm}{}^{[\mathrm{CB}]}$ used on the component level in the article [6],

$$\hat{F}^{+}_{nm\,[\mathrm{CB}]} = e_n{}^b e_m{}^a F^{+}_{ba\,[\mathrm{CB}]}\big| \,, \qquad \hat{F}^{-}_{nm}{}^{[\mathrm{CB}]} = e_n{}^b e_m{}^a F^{-}_{ba}{}^{[\mathrm{CB}]}\big| \,. \tag{2.9}$$

As for $\mathcal{A} =^{\mathrm{A}}_{\alpha}$ and $\mathcal{A} =^{\dot\alpha}_{\mathrm{A}}$ we have the expression of the covariant field strength of the gravitini

$$\begin{aligned}
T_{cb}{}^{\alpha}_{\mathrm{A}}\big| &= e_b{}^m e_c{}^n \mathcal{D}_{[n}\psi_{m]}{}^{\alpha}_{\mathrm{A}} - 4\mathrm{i} e_{[b}{}^n F_{c]a\,[\mathrm{BA}]}\big| \left(\bar\psi_n{}^{\mathrm{B}}\bar\sigma^a\right)^{\alpha} + \frac{1}{4} e_b{}^m e_c{}^n \left(\bar\psi_n{}^{\mathrm{C}}\bar\psi_m{}^{\mathrm{B}}\right)\lambda_{[\mathrm{CBA}]}{}^{\alpha} - \\
&\quad - \frac{\mathrm{i}}{16} e_{[b}{}^n (\psi_{n\mathrm{B}}\sigma_{c]}\bar\sigma_a)^{\alpha}\left(\lambda_{[\mathrm{AEF}]}\sigma^a\lambda^{[\mathrm{BEF}]}\right) + \frac{\mathrm{i}}{48} e_{[b}{}^n (\psi_{n\mathrm{A}}\sigma_{c]a})^{\alpha}\left(\lambda_{[\mathrm{EFG}]}\sigma^a\lambda^{[\mathrm{EFG}]}\right), \\
T_{cb}{}^{\mathrm{A}}_{\dot\alpha}\big| &= e_b{}^m e_c{}^n \mathcal{D}_{[n}\bar\psi_{m]}{}^{\mathrm{A}}_{\dot\alpha} - 4\mathrm{i} e_{[b}{}^n F_{c]a}{}^{[\mathrm{BA}]}\big|(\psi_{n\mathrm{B}}\sigma^a)_{\dot\alpha} + \frac{1}{4} e_b{}^m e_c{}^n (\psi_{n\mathrm{C}}\psi_{m\mathrm{B}})\lambda^{[\mathrm{CBA}]}{}_{\dot\alpha} + \\
&\quad + \frac{\mathrm{i}}{16} e_{[b}{}^n \left(\bar\psi_n{}^{\mathrm{B}}\bar\sigma_{c]}\sigma_a\right)_{\dot\alpha}\left(\lambda_{[\mathrm{BEF}]}\sigma^a\lambda^{[\mathrm{AEF}]}\right) - \frac{\mathrm{i}}{48} e_{[b}{}^n \left(\bar\psi_n{}^{\mathrm{A}}\bar\sigma_{c]a}\right)_{\dot\alpha}\left(\lambda_{[\mathrm{EFG}]}\sigma^a\lambda^{[\mathrm{EFG}]}\right).
\end{aligned}$$

Defining $\hat\Psi_{nm}{}^{\alpha}_{\mathrm{A}} = e_n{}^c e_m{}^b T_{cb}{}^{\alpha}_{\mathrm{A}}\big|$ and respectively $\hat\Psi_{nm}{}^{\mathrm{A}}_{\dot\alpha} = e_n{}^c e_m{}^b T_{cb}{}^{\mathrm{A}}_{\dot\alpha}\big|$, one obtains the supercovariant field strength of the gravitini used in component formulations.

Finally, for the field strength of the scalars Ω_a, we can use the definition of the double projection on the one-form $\Omega = (DT)S$,

$$\Omega\| = \mathrm{d}x^m \Omega_m = e^{\mathcal{A}}\Omega_{\mathcal{A}}\big| \,, \tag{2.10}$$

and obtain

$$\Omega_a\big| = e_a{}^m \left[\Omega_m - \frac{1}{2}\psi_m{}^{\alpha}_{\mathrm{A}}\Omega^{\mathrm{A}}_{\alpha}\big| - \frac{1}{2}\bar\psi_m{}^{\mathrm{A}}_{\dot\alpha}\Omega^{\dot\alpha}_{\mathrm{A}}\big|\right], \tag{2.11}$$

with $\Omega_m = (\mathcal{D}_m\mathcal{T})\mathcal{S}$ the ordinary field strength of the scalars. Then, using the expressions of the matrix components in (A.10), (A.11), we obtain for the two off-diagonal blocks

$$\Omega_a{}^{[\mathrm{DCBA}]}\big| = e_a{}^m \left[\Omega_m{}^{[\mathrm{DCBA}]} + \frac{\eta}{4!}\varepsilon^{\mathrm{DCBAEFGH}}\left(\psi_{m\mathrm{E}}\lambda_{[\mathrm{FGH}]}\right) + \left(\bar\psi_m{}^{[\mathrm{D}}\lambda^{[\mathrm{CBA}]]}\right)\right], \tag{2.12}$$

$$\Omega_{a\,[\mathrm{DCBA}]}\big| = e_a{}^m \left[\Omega_{m\,[\mathrm{DCBA}]} + \left(\psi_{m[\mathrm{D}}\lambda_{[\mathrm{CBA}]]}\right) + \frac{\bar\eta}{4!}\varepsilon_{\mathrm{DCBAEFGH}}\left(\bar\psi_m{}^{\mathrm{E}}\lambda^{[\mathrm{FGH}]}\right)\right], \tag{2.13}$$

and the supercovariant field strength of the scalars is defined as $\hat\Omega_m = e_m{}^a\Omega_a\big|$.

We are now ready to explicite supergravity transformations of the component fields and compare these transformations with those found on the component level [5, 6].

3. Supergravity transformations

Recall that one of the fundamental advantages of the geometrical description in central charge superspace is that space-time diffeomorphisms, supersymmetry and gauge transformations are treated on the same footing as superspace diffeomorphisms. Given that component fields were identified in the supervielbein and in torsion components and also that we know how such geometrical objects transform under supergravity transformations, it is a straightforward exercice to write down how these transformations act on components.

Let us begin with the component fields identified in the frame (1.1): the graviton, the gravitini and graviphotons. Their supergravity transformations can be read out of

$$\delta^{\rm WZ}_\xi E_m{}^{\mathcal{A}}| = \mathcal{D}_m \xi^{\mathcal{A}}| + E_m{}^{\mathcal{B}} \xi^{\mathcal{C}} T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}}| \tag{3.1}$$

and choosing for $\mathcal{A}$ either vector, spinorial or respectively central charge indices. As for the scalars and spinor fields, identified in torsion components, that is as the lowest superfield component of supercovariant fields V, their supergravity transformation is simply

$$\delta^{\rm WZ}_\xi V| = \mathcal{L}_\xi V| = \xi^{\mathcal{A}} \mathcal{D}_{\mathcal{A}} V| \,. \tag{3.2}$$

3.1 Supersymmetry transformations

The supersymmetry transformations are supergravity transformations with only spinorial non-zero parameters

$$\xi^{\mathcal{A}} = \left(0, \xi^\alpha_{\rm A}, \bar\xi^{\rm A}_{\dot\alpha}, 0\right) . \tag{3.3}$$

Therefore, using the general expressions (3.1), (3.2) as well as the expressions of torsion components and spinorial derivatives of the basic fields summed up in the appendix, we have the following component transformation laws.

For the graviton, gravitini and graviphotons we have

$$\begin{aligned}
\delta^{\rm WZ}_\xi e_m{}^a &= {\rm i}\left(\xi_{\rm A}\sigma^a\bar\psi_m{}^{\rm A} + \bar\xi^{\rm A}\bar\sigma^a\psi_{m{\rm A}}\right), && (3.4)\\
\frac{1}{2}\delta^{\rm WZ}_\xi \psi_{m{\rm A}}^{\alpha} &= \mathcal{D}_m\xi^\alpha_{\rm A} - \frac{1}{2}(\bar\xi^{\rm C}\bar\psi_m{}^{\rm B})\lambda_{[{\rm CBA}]}{}^\alpha + \\
&\quad +4{\rm i}(\bar\xi^{\rm C}\bar\sigma^n)^\alpha \hat F^+_{nm\,[{\rm CA}]} + \frac{2{\rm i}\eta}{(4!)^2}\varepsilon_{{\rm ACF_1\ldots F_6}}(\bar\xi^{\rm C}\lambda^{[{\rm F_1F_2F_3}]})(\lambda^{[{\rm F_4F_5F_6}]}\bar\sigma_m)^\alpha - \\
&\quad -\frac{\rm i}{16}(\xi_{\rm F}\sigma_m\bar\sigma_n)^\alpha\left(\lambda_{[{\rm ACB}]}\sigma^n\lambda^{[{\rm FCB}]}\right) + \frac{\rm i}{48}(\xi_{\rm A}\sigma_{mn})^\alpha\left(\lambda_{[{\rm FCB}]}\sigma^n\lambda^{[{\rm FCB}]}\right), && (3.5)\\
\frac{1}{2}\delta^{\rm WZ}_\xi \bar\psi_{m}{}^{\rm A}_{\dot\alpha} &= \mathcal{D}_m\bar\xi^{\rm A}_{\dot\alpha} - \frac{1}{2}(\xi_{\rm C}\psi_{m{\rm B}})\lambda^{[{\rm CBA}]}{}_{\dot\alpha} + \\
&\quad +4{\rm i}(\xi_{\rm C}\sigma^n)_{\dot\alpha}\hat F^-_{nm}{}^{[{\rm CA}]} + \frac{2{\rm i}\bar\eta}{(4!)^2}\varepsilon^{{\rm ACF_1\ldots F_6}}(\xi_{\rm C}\lambda_{[{\rm F_1F_2F_3}]})(\lambda_{[{\rm F_4F_5F_6}]}\sigma_m)_{\dot\alpha} + \\
&\quad +\frac{\rm i}{16}\left(\bar\xi^{\rm F}\bar\sigma_m\sigma_n\right)_{\dot\alpha}\left(\lambda_{[{\rm FCB}]}\sigma^n\lambda^{[{\rm ACB}]}\right) - \frac{\rm i}{48}\left(\bar\xi^{\rm A}\bar\sigma_{mn}\right)_{\dot\alpha}\left(\lambda_{[{\rm FCB}]}\sigma^n\lambda^{[{\rm FCB}]}\right), && (3.6)\\
\delta^{\rm WZ}_\xi v_m{}^{\mathbf{u}} &= -\frac{1}{2}\left[(\xi_{\rm B}\psi_{m{\rm A}}) + \frac{\rm i}{4}\left(\bar\xi^{\rm C}\bar\sigma_m\lambda_{[{\rm CBA}]}\right)\right]\mathcal{T}^{[{\rm BA}]\mathbf{u}} - \\
&\quad -\frac{1}{2}\left[\left(\bar\xi^{\rm B}\bar\psi_m{}^{\rm A}\right) + \frac{\rm i}{4}\left(\xi_{\rm C}\sigma_m\lambda^{[{\rm CBA}]}\right)\right]\mathcal{T}_{[{\rm BA}]}{}^{\mathbf{u}} . && (3.7)
\end{aligned}$$

Using the spinorial derivatives (A.31)–(A.34) of the gravigini superfields we obtain for the helicity 1/2 fields the transformation law

$$\begin{aligned}
\delta^{\rm WZ}_\xi \lambda_{[{\rm CBA}]}{}^\alpha &= 4\delta^{\rm DEF}_{\rm CBA}(\xi_{\rm D}\sigma^{mn})^\alpha \hat F^+_{mn\,[{\rm EF}]} + 4{\rm i}\left(\bar\xi^{\rm D}\bar\sigma^m\right)^\alpha\hat\Omega_{m\,[{\rm DCBA}]} - \\
&\quad -\frac{\bar\eta}{4!}\varepsilon_{{\rm CBAE_1\ldots E_5}}\xi^\alpha_{\rm D}\left(\lambda^{[{\rm DE_1E_2}]}\lambda^{[{\rm E_3E_4E_5}]}\right), && (3.8)\\
\delta^{\rm WZ}_\xi \lambda^{[{\rm CBA}]}{}_{\dot\alpha} &= 4\delta^{\rm CBA}_{\rm DEF}\left(\bar\xi^{\rm D}\bar\sigma^{mn}\right)_{\dot\alpha}\hat F^-_{mn}{}^{[{\rm EF}]} + 4{\rm i}(\xi_{\rm D}\sigma^m)_{\dot\alpha}\hat\Omega_m{}^{[{\rm DCBA}]} - \\
&\quad -\frac{\eta}{4!}\varepsilon^{{\rm CBAE_1\ldots E_5}}\bar\xi^{\rm D}_{\dot\alpha}\left(\lambda_{[{\rm DE_1E_2}]}\lambda_{[{\rm E_3E_4E_5}]}\right). && (3.9)
\end{aligned}$$

As for the supersymmetry transformations of the scalar fields, one obtains

$$\delta_\xi^{\mathrm{WZ}} T| = \mathcal{L}_\xi T| = (\imath_\xi \Omega)\, T| \,, \tag{3.10}$$

that is, just a rotation by a matrix Σ, which is an element of the orthogonal complement of the Lie algebra $su(8)$ in the Lie algebra of $E_{7(+7)}$,

$$\delta_\xi^{\mathrm{WZ}} \mathcal{T} = \Sigma\, \mathcal{T} \,, \quad \text{with} \quad \Sigma = \imath_\xi \Omega| = \begin{pmatrix} 0 & \Sigma^{[\mathrm{DCBA}]} \\ \Sigma_{[\mathrm{DCBA}]} & 0 \end{pmatrix}, \tag{3.11}$$

or in matrix components,

$$\delta_\xi^{\mathrm{WZ}} \mathcal{T}^{[\mathrm{DC}]\mathbf{u}} = \Sigma^{[\mathrm{DCBA}]} \mathcal{T}_{[\mathrm{BA}]}{}^{\mathbf{u}} \,, \qquad \delta_\xi^{\mathrm{WZ}} \mathcal{T}_{[\mathrm{DC}]}{}^{\mathbf{u}} = \Sigma_{[\mathrm{DCBA}]} \mathcal{T}^{[\mathrm{BA}]\mathbf{u}} \,. \tag{3.12}$$

Here, of course, the objects $\Sigma^{[\mathrm{DCBA}]}$ and $\Sigma_{[\mathrm{DCBA}]}$ are related by the duality relation (A.9) and they are computed using the component expressions (A.10) and (A.11) of $\Omega_\alpha^{\mathrm{A}}$ and $\Omega_{\mathrm{A}}^{\dot\alpha}$:

$$\Sigma^{[\mathrm{DCBA}]} = -2 \left[\frac{\eta}{4!} \varepsilon^{\mathrm{DCBAEFGH}} \left(\xi_{\mathrm{E}} \lambda_{[\mathrm{FGH}]} \right) + \left(\bar\xi^{[\mathrm{D}} \lambda^{[\mathrm{CBA}]]} \right) \right], \tag{3.13}$$

$$\Sigma_{[\mathrm{DCBA}]} = -2 \left[\left(\xi_{[\mathrm{D}} \lambda_{[\mathrm{CBA}]]} \right) + \frac{\bar\eta}{4!} \varepsilon_{\mathrm{DCBAEFGH}} \left(\bar\xi^{\mathrm{E}} \lambda^{[\mathrm{FGH}]} \right) \right]. \tag{3.14}$$

Finally, we can also notice, that as pointed out on the component level in the article [6], the supersymmetry transformations of the SU(8) connection can also be given in a simple way using the above defined field-dependent parameters Σ. Indeed, using the expression (A.25) of the SU(8) curvature, we have

$$\delta_\xi^{\mathrm{WZ}} \Phi^{\mathrm{B}}{}_{\mathrm{A}} = \imath_\xi R^{\mathrm{B}}{}_{\mathrm{A}} = \frac{1}{3} \left[\Sigma^{[\mathrm{BEFG}]} \Omega_{[\mathrm{AEFG}]} - \Sigma_{[\mathrm{AEFG}]} \Omega^{[\mathrm{BEFG}]} \right], \tag{3.15}$$

while on the component level,

$$\delta_\xi^{\mathrm{WZ}} \Phi_m{}^{\mathrm{B}}{}_{\mathrm{A}} = \frac{1}{3} \left[\Sigma^{[\mathrm{BEFG}]} \Omega_{m[\mathrm{AEFG}]} - \Sigma_{[\mathrm{AEFG}]} \Omega_m{}^{[\mathrm{BEFG}]} \right]. \tag{3.16}$$

These transformation laws are in perfect concordance with those found at the component level in [6].

3.2 Central charge transformations

In the geometrical description central charge transformations are just supergravity transformations in the direction of central charge coordinates, that is, with parameters

$$\zeta^{\mathcal{A}} = (0, 0, 0, \zeta^{\mathbf{u}}) \,. \tag{3.17}$$

Therefore, we can use the general formulas (3.1) and (3.2) with these parameters in order to give these transformations. However, as a consequence of $T_{\mathbf{z}\mathcal{B}}{}^{\mathcal{A}} = 0$ as well as of the fact that the central charge derivative of all torsion components vanish, $\mathcal{D}_{\mathbf{z}} T_{\mathcal{CB}}{}^{\mathcal{A}} = 0$, all component fields but the graviphotons transform trivially under central charge transformations. The transformation of the graviphotons is simply

$$\delta_\zeta^{\mathrm{WZ}} v_m{}^{\mathbf{u}} = \mathcal{D}_m \zeta^{\mathbf{u}} \,, \tag{3.18}$$

and reflects the fact that they are the gauge fields corresponding to the central charge transformations.

Also, it was noticed in [6], that a gauge transformation with a scalar-dependent parameter appears in the commutator of two supersymmetry transformations. This feature appears naturally in our approach, since one handles with central charge transformations which are present in the algebra of supergravity transformations with a parameter which depends on the 0 dimensional torsion components, where the scalars were identified. In order to see this in detail, recall that the commutator of two supergravity transformations acting on the frame is

$$\left[\delta_{\xi}^{\mathrm{WZ}}, \delta_{\eta}^{\mathrm{WZ}}\right] E^{\mathcal{A}} \;=\; \delta_{[\xi,\eta]}^{\mathrm{WZ}} E^{\mathcal{A}} - E^{\mathcal{B}} \imath_{\xi} \imath_{\eta} R_{\mathcal{B}}{}^{\mathcal{A}}\,, \tag{3.19}$$

with

$$[\xi,\eta]^{\mathcal{A}} \;=\; \xi^{\mathcal{B}} \eta^{\mathcal{C}} T_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}} + \xi^{\mathcal{B}}(\mathcal{D}_{\mathcal{B}} \eta^{\mathcal{A}}) - \eta^{\mathcal{B}}(\mathcal{D}_{\mathcal{B}} \xi^{\mathcal{A}}). \tag{3.20}$$

Then choosing for instance the only non-zero parameters $\xi_{\mathrm{A}}^{\alpha}|$ and $\eta_{\mathrm{A}}^{\alpha}|$, the commutator on a graviphoton becomes

$$\begin{aligned}\left[\delta_{\xi}^{\mathrm{WZ}}, \delta_{\eta}^{\mathrm{WZ}}\right] v_m{}^{\mathbf{u}} = \mathcal{D}_m\Big(\,(\xi_{\mathrm{B}}\eta_{\mathrm{A}})\mathcal{T}^{[\mathrm{BA}]\mathbf{u}}\Big) - \frac{\mathrm{i}}{8}(\xi_{\mathrm{B}}\eta_{\mathrm{A}})(\lambda^{[\mathrm{BAF}]}\bar{\sigma}_m\lambda_{[\mathrm{FDC}]})\mathcal{T}^{[\mathrm{DC}]\mathbf{u}} - \\ -\frac{1}{2}\,(\xi_{\mathrm{B}}\eta_{\mathrm{A}})\left(\lambda^{[\mathrm{BAD}]}\bar{\psi}_m{}^{\mathrm{C}}\right)\mathcal{T}_{[\mathrm{DC}]}{}^{\mathbf{u}}\,,\end{aligned} \tag{3.21}$$

and indeed, it contains a central charge transformation (3.18) with parameter $(\xi_{\mathrm{B}}\eta_{\mathrm{A}})\mathcal{T}^{[\mathrm{BA}]\mathbf{u}}$ depending on scalars as well as a supersymmetry transformation (3.7) with parameter $(\xi_{\mathrm{B}}\eta_{\mathrm{A}})\lambda^{[\mathrm{BAF}]}{}_{\dot{\alpha}}$ depending on the helicity 1/2 fields.

4. The equations of motion

The problem of the derivation of field equations of motion without the knowledge of a lagrangian, using considerations on representations of the symmetry group, was considered for a long time [14, 15]. The question is particularly interesting for supersymmetric theories and for this case various approaches have been developed. Here we use the techniques of superspace geometry introduced by Wess and Zumino, which consist in looking to consequences of covariant constraints corresponding to on-shell field content of a representation of the supersymmetry algebra.

The method of deducing the equations of motion for $N \geq 3$ extended supergravity is similar to the case of the $N = 1$ Yang-Mills theory in the sense that the gravigino superfields $T_{[\mathrm{CBA}]\alpha}$, $T^{[\mathrm{CBA}]\dot{\alpha}}$ play an analogous rôle to the gaugino superfields and all equations of motion but those for the graviton and gravitini are contained in their higher superfield components [16]. Therefore, these equations of motion are found by successively acting with spinorial derivatives on the gravigini superfields. This is the approach which was adopted in previous superspace descriptions of the $N = 8$ supergravity in order to derive the *free* equations of motion from the geometry [9, 8]. At the linearized level this approach has been used to show that usual constraints in superspace imply conformal equations of motion for any $N \geq 4$ [17, 18]. Alternatively, one could also just pick out certain Bianchi identities, which give the equations of motion for the component fields. This strategy to obtain equations of motion is outlined in the article [7]. For the purpose of putting in

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evidence free equations of motion of component fields it is sufficient to consider only the linearized version and the calculations are simple. However, one has to consider the full theory if one wants to obtain all the *non-linear terms* which arise in equations of motion derived from a lagrangian in component formalism.

To begin with let us explain in detail how one can derive the equations of motion for the gravigini. First, recall that the spinorial derivatives of the gravigino superfields $T_{[\mathrm{CBA}]\alpha}$ (see the appendix) have for instance the properties

$$\mathcal{D}^{\mathrm{D}\alpha}T_{[\mathrm{CBA}]\alpha} = \frac{\bar{\eta}}{12}\varepsilon_{\mathrm{CBAIJKLM}}\left(T^{[\mathrm{DIJ}]}T^{[\mathrm{KLM}]}\right)\,, \tag{4.1}$$

$$\sum_{\mathrm{DC}}\mathcal{D}^{\dot{\delta}}_{\mathrm{D}}T_{[\mathrm{CBA}]\alpha} = 0\,. \tag{4.2}$$

In order to get the Dirac equation of the 1/2 helicity field, one can just act on the last relation by the spinorial derivative $\mathcal{D}^{\mathrm{E}}_{\varepsilon}$ obtaining

$$\sum_{\mathrm{DC}}\left(\left\{\mathcal{D}^{\mathrm{E}}_{\varepsilon},\mathcal{D}^{\dot{\delta}}_{\mathrm{D}}\right\}T_{[\mathrm{CBA}]\alpha} - \mathcal{D}^{\dot{\delta}}_{\mathrm{D}}\left(\mathcal{D}^{\mathrm{E}}_{\varepsilon}T_{[\mathrm{CBA}]\alpha}\right)\right) = 0\,, \tag{4.3}$$

and take the antisymmetric part of this relation in the indices ε and α. Then, using the algebra of covariant derivatives as well as equation (4.1) and again the expressions of the spinorial derivatives of the gravigini superfields one obtains

$$\begin{aligned}\mathcal{D}^{\alpha\dot{\alpha}}T_{[\mathrm{CBA}]\alpha} = {}& \frac{\mathrm{i}\bar{\eta}}{3}\varepsilon_{\mathrm{CBAEFGHI}}F^{-\,[\mathrm{EF}]}_{ba}\left(\bar{\sigma}^{ba}T^{[\mathrm{GHI}]}\right)^{\dot{\alpha}} + \\ & + \frac{\mathrm{i}}{16}\left(T_{[\mathrm{CBA}]}T_{[\mathrm{EFG}]}\right)T^{[\mathrm{EFG}]\dot{\alpha}} - \frac{\mathrm{i}}{8}\oint_{\mathrm{CBA}}\left(T_{[\mathrm{CEF}]}T_{[\mathrm{BAG}]}\right)T^{[\mathrm{EFG}]\dot{\alpha}}.\end{aligned} \tag{4.4}$$

Alternatively, it is worthwhile to observe that this equation of motion can also be deduced from the Bianchi identity $\left({}^{\dot{\delta}}_{\mathrm{D}}\,{}^{\dot{\beta}}_{c}\,{}^{\alpha}_{\mathrm{BA}}\right)_{T}$ of dimension 3/2 using the expressions of torsion components and of spinorial derivatives of lower canonical dimension.

The conjugate Dirac equation can be obtained in an analogous way:

$$\begin{aligned}\mathcal{D}_{\alpha\dot{\alpha}}T^{[\mathrm{CBA}]\dot{\alpha}} = {}& \frac{\mathrm{i}\eta}{3}\varepsilon^{\mathrm{CBAEFGHI}}F^{+}_{ba\,[\mathrm{EF}]}\left(\sigma^{ba}T_{[\mathrm{GHI}]}\right)_{\alpha} + \\ & + \frac{\mathrm{i}}{16}\left(T^{[\mathrm{CBA}]}T^{[\mathrm{EFG}]}\right)T_{[\mathrm{EFG}]\alpha} - \frac{\mathrm{i}}{8}\oint^{\mathrm{CBA}}\left(T^{[\mathrm{CEF}]}T^{[\mathrm{BAG}]}\right)T_{[\mathrm{EFG}]\alpha}.\end{aligned} \tag{4.5}$$

Moreover, the equations of motion for the graviphotons and those of the scalars can be deduced by further acting with covariant spinorial derivatives on the equations of motion (4.4) and (4.5) of the gravigini fields as follows.

The trace part in the SU(8) indices of the spinorial derivative $\mathcal{D}^{\mathrm{D}}_{\delta}$ of the Dirac equation (4.4) gives the equations of motion for the self-dual field strength $F^{+}_{ba\,[\mathrm{BA}]}$ of the graviphotons,

$$\begin{aligned}\mathcal{D}_{f}F^{+fa}{}_{[\mathrm{BA}]} = {}& -\mathcal{D}_{f}\left[\frac{\bar{\eta}}{2(4!)^{2}}\varepsilon_{\mathrm{BAF_1\ldots F_6}}\left(T^{[\mathrm{F_1F_2F_3}]}\bar{\sigma}^{fa}T^{[\mathrm{F_4F_5F_6}]}\right)\right] + \\ & + \Omega_{f\,[\mathrm{BAIJ}]}\left[F^{-\,fa\,[\mathrm{IJ}]} + \frac{\eta}{2(4!)^{2}}\varepsilon^{\mathrm{IJF_1\ldots F_6}}\left(T_{[\mathrm{F_1F_2F_3}]}\sigma^{fa}T_{[\mathrm{F_4F_5F_6}]}\right)\right] -\end{aligned}$$

$$-\frac{1}{4(4!)}\left(T_{[\mathrm{FBA}]}\sigma^f\bar\sigma^a T_{[\mathrm{IJK}]}\right)\Omega_f{}^{[\mathrm{FIJK}]}+\frac{\mathrm{i}}{16}\left(T_{[\mathrm{FBA}]}\sigma^{fg}\sigma^a T^{[\mathrm{FIJ}]}\right)F^+_{fg\,[\mathrm{IJ}]}+$$
$$+\frac{\mathrm{i}\bar\eta}{2(4!)^3}\varepsilon_{\mathrm{IJKLMEFG}}\left(T^{[\mathrm{DLM}]}T^{[\mathrm{EFG}]}\right)\left(T_{[\mathrm{DBA}]}\sigma^a T^{[\mathrm{IJK}]}\right)\,. \tag{4.6}$$

The conjugate field equations for $F^-_{ba\,[\mathrm{BA}]}$ can be obtained in an analogous way. Due to the self-duality properties of these field strengths, these equations of motion are related to their Bianchi identities as follows

$$\frac{\mathrm{i}}{2}\varepsilon^{facb}\mathcal{D}_f F^+_{cb\,[\mathrm{BA}]}=\mathcal{D}_f F^{+\,fa}{}_{[\mathrm{BA}]}\,,\qquad \frac{\mathrm{i}}{2}\varepsilon^{facb}\mathcal{D}_f F^{-\,[\mathrm{BA}]}_{cb}=-\mathcal{D}_f F^{-\,fa\,[\mathrm{BA}]}\,. \tag{4.7}$$

Again, it is worthwhile to notice that the Bianchi identities of the graviphotons can be found directly among the super Bianchi identities, namely they are the identities $({}_{dcb}{}^{\mathbf{u}})_T$. As a matter of fact the form given here can be recovered from the Bianchi identity $({}_{dcb}{}^{\mathbf{u}})_T$ after converting the central charge index into SU(8) ones by multiplying by the scalar matrices $S_{\mathbf{u}}{}^{[\mathrm{BA}]}$ and $S_{\mathbf{u}[\mathrm{BA}]}$, as well as using the equations of motion for the gravitini fields presented below in equations (4.10) and (4.11).

On the other hand, take the spinorial derivative $\mathcal{D}^{\dot\delta}_{\mathrm{D}}$ of the same Dirac equation (4.4) and after commuting on the left-hand-side this derivative with that of the space-time one, take the totally antisymmetric part in the SU(8) indices. Now, the antisymmetric part in the spinorial indices $\dot\delta$ and $\dot\alpha$ gives the equations of motion for the scalars written for the field strength component $\Omega_{f\,[\mathrm{DCBA}]}$

$$\begin{aligned}\mathcal{D}^f\Omega_{f\,[\mathrm{DCBA}]}=&-2\delta^{\mathrm{E}_1\ldots\mathrm{E}_4}_{\mathrm{DCBA}}F^{+ba}{}_{[\mathrm{E}_1\mathrm{E}_2]}F^+_{ba\,[\mathrm{E}_3\mathrm{E}_4]}-2\bar\eta\varepsilon_{\mathrm{DCBAE}_1\ldots\mathrm{E}_4}F^{-\,ba\,[\mathrm{E}_1\mathrm{E}_2]}F^{-\,[\mathrm{E}_3\mathrm{E}_4]}_{ba}+\\&+\frac{\mathrm{i}}{2(3!)}\Omega_{f\mathrm{DCBA}}\left(T_{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\sigma^f T^{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\right)-\\&-\frac{\mathrm{i}}{2(4!)}\delta^{\mathrm{E}_1\ldots\mathrm{E}_4}_{\mathrm{DCBA}}\Omega_{f\,[\mathrm{E}_1\mathrm{E}_2\mathrm{E}_3\mathrm{F}_1]}\left(T_{[\mathrm{E}_4\mathrm{F}_2\mathrm{F}_3]}\sigma^f T^{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\right)-\\&-\frac{\mathrm{i}}{32}\delta^{\mathrm{E}_1\ldots\mathrm{E}_4}_{\mathrm{DCBA}}\Omega_{f\,[\mathrm{E}_1\mathrm{E}_2\mathrm{F}_1\mathrm{F}_2]}\left(T_{[\mathrm{E}_3\mathrm{E}_4\mathrm{F}_3]}\sigma^f T^{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\right)+\\&+\frac{\mathrm{i}}{2(3!)^2}\delta^{\mathrm{E}_1\ldots\mathrm{E}_4}_{\mathrm{DCBA}}\Omega_{f\,[\mathrm{E}_1\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\left(T_{[\mathrm{E}_2\mathrm{E}_3\mathrm{E}_4]}\sigma^f T^{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\right)+\\&+\frac{5\eta}{2(3!)(4!)^3}\delta^{\mathrm{E}_1\ldots\mathrm{E}_4}_{\mathrm{DCBA}}\varepsilon^{\mathrm{F}_1\ldots\mathrm{F}_8}\left(T_{[\mathrm{E}_1\mathrm{E}_2\mathrm{E}_3]}T_{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\right)\left(T_{[\mathrm{E}_4\mathrm{F}_4\mathrm{F}_5]}T_{[\mathrm{F}_6\mathrm{F}_7\mathrm{F}_8]}\right)-\\&-\frac{\eta}{256(4!)}\delta^{\mathrm{E}_1\ldots\mathrm{E}_4}_{\mathrm{DCBA}}\varepsilon^{\mathrm{F}_1\ldots\mathrm{F}_8}\left(T_{[\mathrm{E}_1\mathrm{E}_2\mathrm{F}_1]}T_{[\mathrm{E}_3\mathrm{F}_2\mathrm{F}_3]}\right)\left(T_{[\mathrm{E}_4\mathrm{F}_4\mathrm{F}_5]}T_{[\mathrm{F}_6\mathrm{F}_7\mathrm{F}_8]}\right)+\\&+\frac{5\bar\eta^2}{2(3!)(4!)^3}\varepsilon_{\mathrm{DCBAE}_1\ldots\mathrm{E}_4}\varepsilon_{\mathrm{F}_1\ldots\mathrm{F}_8}\left(T^{[\mathrm{E}_1\mathrm{E}_2\mathrm{E}_3]}T^{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\right)\left(T^{[\mathrm{E}_4\mathrm{F}_4\mathrm{F}_5]}T^{[\mathrm{F}_6\mathrm{F}_7\mathrm{F}_8]}\right)-\\&-\frac{\bar\eta^2}{256(4!)}\varepsilon_{\mathrm{DCBAE}_1\ldots\mathrm{E}_4}\varepsilon_{\mathrm{F}_1\ldots\mathrm{F}_8}\left(T^{[\mathrm{E}_1\mathrm{E}_2\mathrm{F}_1]}T^{[\mathrm{E}_3\mathrm{F}_2\mathrm{F}_3]}\right)\left(T^{[\mathrm{E}_4\mathrm{F}_4\mathrm{F}_5]}T^{[\mathrm{F}_6\mathrm{F}_7\mathrm{F}_8]}\right).\end{aligned} \tag{4.8}$$

The equations of motion for the field strength component $\Omega_f{}^{[\mathrm{DCBA}]}$ can be calculated in an analogous way from the Dirac equation (4.5) and one can check that of course, the two equations are related by η-duality.

The symmetric part in the indices $\dot{\delta}$ and $\dot{\alpha}$ is the anti-self-dual part of the Bianchi identity for the scalars,

$$\mathcal{D}_d \Omega_c^{[\mathrm{DCBA}]} - \mathcal{D}_c \Omega_d^{[\mathrm{DCBA}]} = -32 \left[\left(T_{dc}{}^{[\mathrm{D}} T^{\mathrm{CBA}]]} \right) + \frac{\eta}{4!} \epsilon^{\mathrm{DCBAFIJK}} \left(T_{dc\mathrm{F}} T g_{[\mathrm{IJK}]} \right) \right], \qquad (4.9)$$

which can be deduced also from the off-diagonal part of the superspace Bianchi identities of the one-form Ω.

Unlike the equations of motion presented above, in Poincaré supergravity the equations of motion for the gravitini and graviton cannot be obtained by acting again by spinorial derivatives on the equations of motion obtained so far. One could obtain this way only the Bianchi identities for these fields (which can be obtained also in a much direct way: the Bianchi identities of the gravitini are exactly the superspace Bianchi identities $\left({}_{dcb}{}^{\alpha}_{\mathrm{A}}\right)_T$ and $\left({}_{dcb}{}^{\mathrm{A}}_{\dot{\alpha}}\right)_T$, since the Bianchi identity for the graviton is just the dcb component of the Bianchi identity for the Lorentz curvature). The equations of motion for the gravitini and of the graviton are directly given by the superspace Bianchi identities.

For example, the Bianchi identities $\begin{pmatrix} \mathrm{D} & \mathrm{B}\alpha \\ \delta & c\beta\,\mathrm{A} \end{pmatrix}_T$ and $\begin{pmatrix} \dot{\delta} & \dot{\beta}\,\mathrm{A} \\ \mathrm{D} & c\mathrm{B}\dot{\alpha} \end{pmatrix}_T$ give the irreducible components $T_{(\beta\alpha)}{}^{\alpha}_{\mathrm{A}}$ and $T^{(\dot{\beta}\dot{\alpha})}{}^{\alpha}_{\mathrm{A}}$ of the gravitini field strength $T_{ba}{}^{\alpha}_{\mathrm{A}}$ and respectively the irreducible components $T_{(\beta\alpha)}{}^{\mathrm{A}}_{\dot{\alpha}}$ and $T^{(\dot{\beta}\dot{\alpha})}{}^{\mathrm{A}}_{\dot{\alpha}}$ of the gravitini field strength $T_{ba}{}^{\mathrm{A}}_{\dot{\alpha}}$ as a non-linear function of basic superfields. It turns out that the knowledge of these parts is sufficient to write down the equations of motion of the gravitini fields:

$$\begin{aligned} \varepsilon^{dcba} (\bar{\sigma}_c T_{ba\mathrm{A}})^{\dot{\alpha}} &= \frac{1}{3!} \left(\bar{\sigma}^a \sigma^d T^{[\mathrm{DCB}]} \right)^{\dot{\alpha}} \Omega_{a[\mathrm{DCBA}]} + \mathrm{i} \left(\bar{\sigma}^{ba} \bar{\sigma}^d T_{[\mathrm{CBA}]} \right)^{\dot{\alpha}} \mathcal{F}^{-}_{ba}{}^{[\mathrm{CB}]} - \\ &\quad - \frac{\mathrm{i}\eta}{3(4!)^2} \varepsilon^{\mathrm{F}_1 \ldots \mathrm{F}_8} \left(T_{[\mathrm{AF}_1\mathrm{F}_2]} T_{[\mathrm{F}_3\mathrm{F}_4\mathrm{F}_5]} \right) \left(\bar{\sigma}^d T_{[\mathrm{F}_6\mathrm{F}_7\mathrm{F}_8]} \right)^{\dot{\alpha}}, \qquad (4.10) \\ \varepsilon^{dcba} (\sigma_c T_{ba}{}^{\mathrm{A}})_{\alpha} &= -\frac{1}{3!} \left(\sigma^a \bar{\sigma}^d T_{[\mathrm{DCB}]} \right)_{\alpha} \Omega_a{}^{[\mathrm{DCBA}]} - \mathrm{i} \left(\sigma^{ba} \sigma^d T^{[\mathrm{CBA}]} \right)_{\alpha} \mathcal{F}^{+}_{ba[\mathrm{CB}]} + \\ &\quad + \frac{\mathrm{i}\eta}{3(4!)^2} \varepsilon_{\mathrm{F}_1 \ldots \mathrm{F}_8} \left(T^{[\mathrm{AF}_1\mathrm{F}_2]} T^{[\mathrm{F}_3\mathrm{F}_4\mathrm{F}_5]} \right) \left(\sigma^d T^{[\mathrm{F}_6\mathrm{F}_7\mathrm{F}_8]} \right)_{\alpha}. \qquad (4.11) \end{aligned}$$

Finally, in order to give the equations of motion for the graviton we need the expression of the supercovariant Ricci tensor, $R_{db} = R_{dcba} \eta^{ca}$, which is given (A.23) by the superspace Bianchi identities at canonical dimension 2. The corresponding Ricci scalar, $R = R_{db} \eta^{db}$, is then also determined (A.24) and using the equations of motion for the spinor fields we find that the Einstein equation takes the form

$$\begin{aligned} R_{db} - \frac{1}{2} \eta_{db} R = &-\frac{1}{3!} \left[\Omega_d{}^{[\mathrm{IJKL}]} \Omega_{b[\mathrm{IJKL}]} - \frac{1}{2} \eta_{db} \Omega^{f[\mathrm{IJKL}]} \Omega_{f[\mathrm{IJKL}]} \right] - 32 F^{+f}_{(d}{}_{[\mathrm{IJ}]} F^{-}_{b)f}{}^{[\mathrm{IJ}]} - \\ &-\frac{\mathrm{i}}{4!} \left[\left(T_{[\mathrm{IJK}]} \sigma_{(d} \mathcal{D}_{b)} T^{[\mathrm{IJK}]} \right) + \left(T^{[\mathrm{IJK}]} \bar{\sigma}_{(d} \mathcal{D}_{b)} T_{[\mathrm{IJK}]} \right) \right] + \\ &+\frac{\eta}{3!4!} \varepsilon^{\mathrm{G}_1 \ldots \mathrm{G}_8} \left[F^{+f}_{(d}{}_{[\mathrm{G}_1\mathrm{G}_2]} \left(T_{[\mathrm{G}_3\mathrm{G}_4\mathrm{G}_5]} \sigma_{b)f} T_{[\mathrm{G}_6\mathrm{G}_7\mathrm{G}_8]} \right) - \right. \\ &\qquad \left. -\frac{5}{4} \eta_{db} F^{+ef}{}_{[\mathrm{G}_1\mathrm{G}_2]} \left(T_{[\mathrm{G}_3\mathrm{G}_4\mathrm{G}_5]} \sigma_{ef} T_{[\mathrm{G}_6\mathrm{G}_7\mathrm{G}_8]} \right) \right] + \\ &+\frac{\bar{\eta}}{3!4!} \varepsilon_{\mathrm{G}_1 \ldots \mathrm{G}_8} \left[F^{-f[\mathrm{G}_1\mathrm{G}_2]}_{(d} \left(T^{[\mathrm{G}_3\mathrm{G}_4\mathrm{G}_5]} \bar{\sigma}_{b)f} T^{[\mathrm{G}_6\mathrm{G}_7\mathrm{G}_8]} \right) - \right. \end{aligned}$$

$$
\begin{aligned}
&-\frac{5}{4}\eta_{db}F^{-ef[\mathrm{G_1G_2}]}\left(T_{[\mathrm{G_3G_4G_5}]}\bar{\sigma}_{ef}T^{[\mathrm{G_6G_7G_8}]}\right)\Big]+\\
&+\frac{1}{64(4!)^2}\delta^{\mathrm{G_1\dots G_6}}_{\mathrm{H_1\dots H_6}}\left(T_{[\mathrm{G_1G_2G_3}]}\sigma_d T^{[\mathrm{H_1H_2H_3}]}\right)\left(T_{[\mathrm{G_4G_5G_6}]}\sigma_b T^{[\mathrm{H_4H_5H_6}]}\right)-\\
&-\frac{1}{32(4!)^2}\eta_{db}\delta^{\mathrm{G_1\dots G_5}}_{\mathrm{H_1\dots H_5}}\left(T_{[\mathrm{FG_1G_2}]}T_{[\mathrm{G_3G_4G_5}]}\right)\left(T^{[\mathrm{FH_1H_2}]}T^{[\mathrm{H_3H_4H_5}]}\right)+\\
&+\frac{1}{4(4!)^2}\Big[\left(T_{[\mathrm{IJK}]}\sigma_d T^{[\mathrm{IJK}]}\right)\left(T_{[\mathrm{LMN}]}\sigma_b T^{[\mathrm{LMN}]}\right)-\\
&\qquad -9\left(T_{[\mathrm{GIJ}]}\sigma_d T^{[\mathrm{FIJ}]}\right)\left(T_{[\mathrm{FKL}]}\sigma_b T^{[\mathrm{GKL}]}\right)\Big]-\\
&-\frac{1}{(4!)^2}\eta_{db}\left(T_{[\mathrm{IJK}]}T_{[\mathrm{LMN}]}\right)\Big[\left(T^{[\mathrm{IJK}]}T^{[\mathrm{LMN}]}\right)+\frac{27}{4}\left(T^{[\mathrm{IJL}]}T^{[\mathrm{KMN}]}\right)\Big]\,,
\end{aligned}
\tag{4.12}
$$

where one may recognize in the first two lines of the right-hand-side the usual terms of the energy-momentum tensor corresponding to matter fields: scalar fields, photon fields and spinor fields respectively. The contribution of the gravitini is hidden in the left-hand-side, in the component development [16] of R_{db}.

The Einstein equation completes the ensemble of the equations of motion for the component fields. Fortunately, in the article [6], which contains a detailed list of the results obtained in the component formulation, most of the equations of motion are given in terms of supercovariant quantities, given in paragraph 2 as functions of the component fields. As a consequence, it is easy to compare the equations of motion deduced in this paragraph from the geometry with the component results of [6], and see that there is a perfect concordance between them.

5. Conclusion

We completed the identification of the $N = 8$ supergravity in the geometry of central charge superspace by the presentation of the deduction from the geometry of both the supergravity transformation laws and the equations of motion for the component fields. Our results obtained from the geometric formulation are in perfect accord with the results in terms of supercovariant quantities obtained in component formulation given in the article [6].

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A. Solution of the Bianchi identities

In this appendix we give the solution of the Bianchi identities, which corresponds to the $N = 8$ supergravity in central charge superspace. All torsion and curvature components are expressed as polynomial functions of the scalar superfields $T^{[\mathrm{CB}]\mathbf{u}}$ and $T_{[\mathrm{CB}]}{}^{\mathbf{u}}$ (or their supercovariant field strength Ω_a), the gravigini superfields $T_{[\mathrm{CBA}]\alpha}$ and $T^{[\mathrm{CBA}]\dot{\alpha}}$, the gravitini

Weyl tensors $\Sigma_{(\gamma\beta\alpha)\mathrm{A}}$ and $\Sigma^{(\dot\gamma\dot\beta\dot\alpha)\mathrm{A}}$, the usual Weyl tensors $V_{(\delta\gamma\beta\alpha)}$ and $V_{(\dot\delta\dot\gamma\dot\beta\dot\alpha)}$, as well as the super-covariant field strength of the graviphotons, identified in the torsion component $T_{cb}{}^{\mathbf{u}}$, denoted also $F_{cb}{}^{\mathbf{u}}$.

The scalar superfields $T^{[\mathrm{CB}]\mathbf{u}}$ and $T_{[\mathrm{CB}]}{}^{\mathbf{u}}$ can be gathered in the 56×56 matrix T having its inverse S,

$$T \doteq \begin{pmatrix} T^{[\mathrm{DC}]\mathbf{u}} \\ T_{[\mathrm{DC}]}{}^{\mathbf{u}} \end{pmatrix}, \qquad S \doteq \begin{pmatrix} S_{\mathbf{u}[\mathrm{DC}]} & S_{\mathbf{u}}{}^{[\mathrm{DC}]} \end{pmatrix}, \tag{A.1}$$

$$TS = \mathbf{1}_{56} = \begin{pmatrix} \frac{1}{2}\delta^{\mathrm{DC}}_{\mathrm{BA}} & 0 \\ 0 & \frac{1}{2}\delta^{\mathrm{BA}}_{\mathrm{DC}} \end{pmatrix}, \qquad ST = \mathbf{1}_{56} = (\delta^{\mathbf{z}}_{\mathbf{u}})\,. \tag{A.2}$$

In fact, these matrices serve as converters between the central charge basis (indices $\mathbf{u}$) and the SU(8) basis in the antisymmetric representation (indices $_{[\mathrm{DC}]}$ and $^{[\mathrm{DC}]}$). For instance all object $X^{\mathbf{u}}$ can be converted in the SU(8) basis using the matrix S:

$$\begin{pmatrix} X_{[\mathrm{DC}]} & X^{[\mathrm{DC}]} \end{pmatrix} = X^{\mathbf{u}} \begin{pmatrix} S_{\mathbf{u}[\mathrm{DC}]} & S_{\mathbf{u}}{}^{[\mathrm{DC}]} \end{pmatrix}. \tag{A.3}$$

Inversely, one can come back to the central charge basis using

$$X^{\mathbf{u}} = \begin{pmatrix} X_{[\mathrm{DC}]} & X^{[\mathrm{DC}]} \end{pmatrix} \begin{pmatrix} T^{[\mathrm{DC}]\mathbf{u}} \\ T_{[\mathrm{DC}]}{}^{\mathbf{u}} \end{pmatrix} = X_{[\mathrm{DC}]}T^{[\mathrm{DC}]\mathbf{u}} + X^{[\mathrm{DC}]}T_{[\mathrm{DC}]}{}^{\mathbf{u}}\,. \tag{A.4}$$

For example, the field strength of the graviphotons can be written in the SU(8) basis:

$$\begin{pmatrix} F_{ba\,[\mathrm{DC}]} & F_{ba}{}^{[\mathrm{DC}]} \end{pmatrix} = F_{ba}{}^{\mathbf{u}} \begin{pmatrix} S_{\mathbf{u}[\mathrm{DC}]} & S_{\mathbf{u}}{}^{[\mathrm{DC}]} \end{pmatrix}. \tag{A.5}$$

As a matter of fact the dynamic part of this field strength corresponds to the self-dual part $F^{+}_{ba\,[\mathrm{BA}]}$ of the component $F_{ba\,[\mathrm{BA}]}$ and the anti-self-dual part $F^{-}_{ba}{}^{[\mathrm{BA}]}$ of the component $F_{ba}{}^{[\mathrm{BA}]}$, while the remaining parts (the anti-self-dual part of the component $F_{ba\,[\mathrm{BA}]}$ and the self-dual part of the component $F_{ba}{}^{[\mathrm{BA}]}$) are given as quadratic terms in the spinor superfields:

$$F_{ba\,[\mathrm{BA}]} = F^{+}_{ba\,[\mathrm{BA}]} - \frac{\eta}{2(4!)^2}\varepsilon_{\mathrm{BAF}_1\ldots\mathrm{F}_6}T^{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\bar\sigma_{ba}T^{[\mathrm{F}_4\mathrm{F}_5\mathrm{F}_6]}\,, \tag{A.6}$$

$$F_{ba}{}^{[\mathrm{BA}]} = F^{-}_{ba}{}^{[\mathrm{BA}]} - \frac{\bar\eta}{2(4!)^2}\varepsilon^{\mathrm{BAF}_1\ldots\mathrm{F}_6}T_{[\mathrm{F}_1\mathrm{F}_2\mathrm{F}_3]}\sigma_{ba}T_{[\mathrm{F}_4\mathrm{F}_5\mathrm{F}_6]}\,. \tag{A.7}$$

Recall also, that the field strength of the scalars Ω_a is the vector component of the super 1-form

$$\Omega = (DT)S = \begin{pmatrix} 0 & \Omega^{[\mathrm{DCBA}]} \\ \Omega_{[\mathrm{DCBA}]} & 0 \end{pmatrix} \tag{A.8}$$

where the forms $\Omega^{[\mathrm{DCBA}]}$ and $\Omega_{[\mathrm{DCBA}]}$ are related by the η-duality relations

$$\Omega^{[\mathrm{HGFE}]} = \frac{\eta}{4!}\varepsilon^{\mathrm{HGFEDCBA}}\Omega_{[\mathrm{DCBA}]}\,, \qquad \Omega_{[\mathrm{HGFE}]} = \frac{\bar\eta}{4!}\varepsilon_{\mathrm{HGFEDCBA}}\Omega^{[\mathrm{DCBA}]}\,, \qquad \eta\bar\eta = 1\,. \tag{A.9}$$

Its components are given by

$$\Omega_\delta^{\rm D} = -\frac{1}{2}\frac{1}{3!}\begin{pmatrix} 0 & \eta\varepsilon^{\rm BAB'A'DEFG}T_{\rm [EFG]\delta} \\ \delta^{\rm DEFG}_{\rm BAB'A'}T_{\rm [EFG]\delta} & 0 \end{pmatrix}, \tag{A.10}$$

$$\Omega_{\rm D}^{\dot\delta} = -\frac{1}{2}\frac{1}{3!}\begin{pmatrix} 0 & \delta^{\rm BAB'A'}_{\rm DEFG}T^{\rm [EFG]\dot\delta} \\ \bar\eta\varepsilon_{\rm BAB'A'DEFG}T^{\rm [EFG]\dot\delta} & 0 \end{pmatrix}, \tag{A.11}$$

$$\Omega_a = -\frac{\rm i}{4}\begin{pmatrix} 0 & P_a{}^{\rm [BAB'A']} \\ P_{a\rm [BAB'A']} & 0 \end{pmatrix}, \tag{A.12}$$

$$\Omega_{\mathbf u} = 0\,. \tag{A.13}$$

The spinorial derivatives of these basic superfields are also expressed as polynomial functions of themselves.

A.1 Non-vanishing torsion components

The list of the non-vanishing torsion components is the following:

$$\lambda_{\gamma\,\rm B}^{\rm C\dot\beta\,a} = -2{\rm i}\delta^{\rm C}_{\rm B}(\sigma^a\epsilon)_\gamma{}^{\dot\beta}\,, \qquad T_{\gamma\,\beta}^{\rm CB\mathbf u} = \epsilon_{\gamma\beta}T^{\rm [CB]\mathbf u}\,, \qquad T^{\dot\gamma\,\dot\beta\,\mathbf u}_{\rm CB} = \epsilon^{\dot\gamma\dot\beta}T_{\rm [CB]}{}^{\mathbf u}\,, \tag{A.14}$$

$$\begin{aligned}
T_{\gamma\,\beta\,\dot\alpha}^{\rm CBA} &= \epsilon_{\beta\gamma}T^{\rm [CBA]}{}_{\dot\alpha}\,, \qquad T^{\dot\gamma\dot\beta\,\alpha}_{\rm CBA} = \epsilon_{\dot\beta\dot\gamma}T_{\rm [CBA]}{}^{\alpha}\,,\\
T_{\gamma\,b}^{\rm C}{}^{\mathbf u} &= -\frac{\rm i}{8}(\sigma_b T^{\rm [CIJ]})_\gamma T_{\rm [IJ]}{}^{\mathbf u}\,, \qquad T^{\dot\gamma}_{{\rm C}b}{}^{\mathbf u} = -\frac{\rm i}{8}(\bar\sigma_b T_{\rm [CIJ]})^{\dot\gamma}T^{\rm [IJ]\mathbf u}\,,\\
T_{\gamma\,b\dot\alpha}^{\rm C\;\;A} &= 4{\rm i}(\sigma^f)_{\gamma\dot\alpha}F_{fb}{}^{\rm [CA]}\,, \qquad T^{\dot\gamma\;\;\alpha}_{{\rm C}b{\rm A}} = 4{\rm i}(\bar\sigma^f)^{\dot\gamma\alpha}F_{fb\rm [CA]}\,,\\
T_{\gamma\,b{\rm A}}^{\rm C\;\;\alpha} &= -\delta^\alpha_\gamma T_b^{\rm C}{}_{\rm A} - 2(\sigma_{bf})_\gamma{}^\alpha U^{f\rm C}{}_{\rm A}\,, \qquad T^{\dot\gamma\;\;{\rm A}}_{{\rm C}b\dot\alpha} = \delta^{\dot\gamma}_{\dot\alpha}T_b^{\rm A}{}_{\rm C} + 2(\bar\sigma_{bf})^{\dot\gamma}{}_{\dot\alpha}U^{f\rm A}{}_{\rm C}\,,
\end{aligned} \tag{A.15}$$

$$\begin{aligned}
T_{cb{\rm A}\alpha} &= -(\epsilon\sigma_{cb})^{\gamma\beta}\Sigma_{(\gamma\beta\alpha){\rm A}} + \frac{\rm i}{18}\left(\delta^{fg}_{cb} - \frac{\rm i}{2}\varepsilon_{cb}{}^{fg}\right)\Omega_{f\rm [ABCD]}\left(\sigma_g T^{\rm [BCD]}\right)_\alpha +\\
&\quad + F^-_{cb}{}^{\rm [CB]}T_{\rm [CBA]\alpha} - \frac{\eta}{4!(3!)^3}\varepsilon^{\rm F_1\ldots F_6CB}(\sigma_{cb}T_{\rm [F_1F_2F_3]})_\alpha\left(T_{\rm [F_4F_5F_6]}T_{\rm [CBA]}\right),
\end{aligned} \tag{A.16}$$

$$\begin{aligned}
T_{cb}{}^{\rm A\dot\alpha} &= -(\epsilon\bar\sigma_{cb})_{\dot\gamma\dot\beta}\Sigma^{(\dot\gamma\dot\beta\dot\alpha){\rm A}} + \frac{\rm i}{18}\left(\delta^{fg}_{cb} + \frac{\rm i}{2}\varepsilon_{cb}{}^{fg}\right)\Omega_f{}^{\rm [ABCD]}\left(\bar\sigma_g T_{\rm [BCD]}\right)^{\dot\alpha} +\\
&\quad + F^+_{cb\,\rm [CB]}T^{\rm [CBA]\dot\alpha} - \frac{\bar\eta}{4!(3!)^3}\varepsilon_{\rm F_1\ldots F_6CB}\left(\bar\sigma_{cb}T^{\rm [F_1F_2F_3]}\right)^{\dot\alpha}\left(T^{\rm [F_4F_5F_6]}T^{\rm [CBA]}\right),
\end{aligned} \tag{A.17}$$

where the superfields $T_b{}^{\rm B}{}_{\rm A}$ and $U_b{}^{\rm B}{}_{\rm A}$ are quadratic terms in the spinor superfields

$$T_b{}^{\rm B}{}_{\rm A} = -\frac{\rm i}{16}T_{\rm [AIJ]}\sigma_b T^{\rm [BIJ]}\,, \qquad U_b{}^{\rm B}{}_{\rm A} = \frac{\rm i}{16}\left(T_{\rm [AIJ]}\sigma_b T^{\rm [BIJ]} - \frac{1}{6}\delta^{\rm B}_{\rm A}T_{\rm [IJK]}\sigma_b T^{\rm [IJK]}\right). \tag{A.18}$$

A.2 Curvature components

Let us begin with giving the list of Lorentz curvature components:

$$R^{\rm DC}_{\delta\,\gamma\,ba} = -16\epsilon_{\delta\gamma}F_{ba}{}^{\rm [DC]}\,, \quad R^{\dot\delta\,\dot\gamma}_{\rm DC\,\it ba} = -16\epsilon^{\dot\delta\dot\gamma}F_{ba\rm [DC]}\,, \quad R^{\rm D\dot\gamma}_{\delta\;{\rm C}ba} = 4\varepsilon_{badc}(\sigma^d\epsilon)_\delta{}^{\dot\gamma}U^{c\rm D}{}_{\rm C}\,. \tag{A.19}$$

$$R^{\rm D}_{\delta\,cba} = {\rm i}\left(\sigma_c T_{ba}{}^{\rm D} - \sigma_a T_{cb}{}^{\rm D} - \sigma_b T_{ac}{}^{\rm D}\right)_\delta\,, \tag{A.20}$$

$$R^{\dot\delta}_{{\rm D}cba} = {\rm i}\left(\bar\sigma_c T_{ba\rm D} - \bar\sigma_a T_{cb\rm D} - \bar\sigma_b T_{ac\rm D}\right)^{\dot\delta}\,, \tag{A.21}$$

$$R_{dcba} = (\epsilon\sigma_{dc})^{\delta\gamma}(\epsilon\sigma_{ba})^{\beta\alpha}V_{(\delta\gamma\beta\alpha)} + (\epsilon\bar{\sigma}_{dc})_{\dot{\delta}\dot{\gamma}}(\epsilon\bar{\sigma}_{ba})_{\dot{\beta}\dot{\alpha}}V^{(\dot{\delta}\dot{\gamma}\dot{\beta}\dot{\alpha})} + \\ +\frac{1}{2}(\eta_{db}R_{ca} - \eta_{da}R_{cb} + \eta_{ca}R_{db} - \eta_{cb}R_{da}) - \frac{1}{6}(\eta_{db}\eta_{ca} - \eta_{da}\eta_{cb})R \qquad \text{(A.22)}$$

with the supercovariant Ricci tensor, $R_{db} = R_{dcba}\eta^{ca}$, given by

$$\begin{aligned} R_{db} = & -\frac{1}{3!}\Omega_d{}^{[\mathrm{IJKL}]}\Omega_{b[\mathrm{IJKL}]} - 32F^{+f}_{(d}{}_{[\mathrm{IJ}]}F^{-}_{b)f}{}^{[\mathrm{IJ}]} + \\ & +\frac{\eta}{3!4!}\varepsilon^{\mathrm{G_1 \ldots G_8}}\Big[F^{+f}_{(d}{}_{[\mathrm{G_1G_2}]}\left(T_{[\mathrm{G_3G_4G_5}]}\sigma_{b)f}T_{[\mathrm{G_6G_7G_8}]}\right) + \\ & \qquad +\frac{1}{4}\eta_{db}F^{+ef}{}_{[\mathrm{G_1G_2}]}\left(T_{[\mathrm{G_3G_4G_5}]}\sigma_{ef}T_{[\mathrm{G_6G_7G_8}]}\right)\Big] + \\ & +\frac{\bar{\eta}}{3!4!}\varepsilon_{\mathrm{G_1 \ldots G_8}}\Big[F^{-}_{(d}{}^{f[\mathrm{G_1G_2}]}\left(T^{[\mathrm{G_3G_4G_5}]}\bar{\sigma}_{b)f}T^{[\mathrm{G_6G_7G_8}]}\right) + \\ & \qquad +\frac{1}{4}\eta_{db}F^{-ef[\mathrm{G_1G_2}]}\left(T_{[\mathrm{G_3G_4G_5}]}\bar{\sigma}_{ef}T^{[\mathrm{G_6G_7G_8}]}\right)\Big] - \\ & -\frac{\mathrm{i}}{4!}\Big[\left(T_{[\mathrm{IJK}]}\sigma_{(d}\mathcal{D}_{b)}T^{[\mathrm{IJK}]}\right) - \frac{1}{4}\eta_{db}\left(T_{[\mathrm{IJK}]}\sigma^f\mathcal{D}_fT^{[\mathrm{IJK}]}\right) - \\ & \quad +\left(T^{[\mathrm{IJK}]}\bar{\sigma}_{(d}\mathcal{D}_{b)}T_{[\mathrm{IJK}]}\right) - \frac{1}{4}\eta_{db}\left(T^{[\mathrm{IJK}]}\bar{\sigma}^f\mathcal{D}_fT_{[\mathrm{IJK}]}\right)\Big] + \\ & +\frac{1}{64(4!)^2}\delta^{\mathrm{G_1 \ldots G_6}}_{\mathrm{H_1 \ldots H_6}}\left(T_{[\mathrm{G_1G_2G_3}]}\sigma_dT^{[\mathrm{H_1H_2H_3}]}\right)\left(T_{[\mathrm{G_4G_5G_6}]}\sigma_bT^{[\mathrm{H_4H_5H_6}]}\right) + \\ & +\frac{1}{32(4!)^2}\eta_{db}\delta^{\mathrm{G_1 \ldots G_5}}_{\mathrm{H_1 \ldots H_5}}\left(T_{[\mathrm{FG_1G_2}]}T_{[\mathrm{G_3G_4G_5}]}\right)\left(T^{[\mathrm{FH_1H_2}]}T^{[\mathrm{H_3H_4H_5}]}\right) + \\ & +\frac{1}{4(4!)^2}\Big[\left(T_{[\mathrm{IJK}]}\sigma_dT^{[\mathrm{IJK}]}\right)\left(T_{[\mathrm{LMN}]}\sigma_bT^{[\mathrm{LMN}]}\right) - \\ & \qquad -9\left(T_{[\mathrm{GIJ}]}\sigma_dT^{[\mathrm{FIJ}]}\right)\left(T_{[\mathrm{FLM}]}\sigma_bT^{[\mathrm{GLM}]}\right)\Big] + \\ & +\frac{1}{2(4!)^2}\eta_{db}\left(T_{[\mathrm{IJK}]}T_{[\mathrm{LMN}]}\right)\Big[\left(T^{[\mathrm{IJK}]}T^{[\mathrm{LMN}]}\right) - 9\left(T^{[\mathrm{IJL}]}T^{[\mathrm{KMN}]}\right)\Big] \end{aligned} \qquad \text{(A.23)}$$

and the corresponding Ricci scalar, $R = R_{db}\eta^{db}$, which is then

$$\begin{aligned} R = & -\frac{1}{3!}\Omega^{f[\mathrm{IJKL}]}\Omega_{f[\mathrm{IJKL}]} + \\ & +\frac{1}{3(4!)}\eta\varepsilon^{\mathrm{G_1 \ldots G_8}}F^{+}_{ca[\mathrm{G_1G_2}]}\left(T_{[\mathrm{G_3G_4G_5}]}\sigma^{ca}T_{[\mathrm{G_6G_7G_8}]}\right) + \\ & +\frac{1}{3(4!)}\bar{\eta}\varepsilon_{\mathrm{G_1 \ldots G_8}}F^{-[\mathrm{G_1G_2}]}_{ca}\left(T^{[\mathrm{G_3G_4G_5}]}\bar{\sigma}^{ca}T^{[\mathrm{G_6G_7G_8}]}\right) + \\ & +\frac{1}{8(4!)^2}\delta^{\mathrm{G_1 \ldots G_5}}_{\mathrm{H_1 \ldots H_5}}\left(T_{[\mathrm{FG_1G_2}]}T_{[\mathrm{G_3G_4G_5}]}\right)\left(T^{[\mathrm{FH_1H_2}]}T^{[\mathrm{H_3H_4H_5}]}\right) + \\ & +\frac{1}{16(4!)}\left(T_{[\mathrm{IJK}]}T_{[\mathrm{LMN}]}\right)\Big[\left(T^{[\mathrm{IJK}]}T^{[\mathrm{LMN}]}\right) - 9\left(T^{[\mathrm{IJL}]}T^{[\mathrm{KMN}]}\right)\Big]. \end{aligned} \qquad \text{(A.24)}$$

The SU(8) curvature is given by

$$R^{\mathrm{B}}{}_{\mathrm{A}} = -\frac{1}{3}\Omega^{[\mathrm{BIJK}]}\Omega_{[\mathrm{AIJK}]}\,. \qquad \text{(A.25)}$$

A.3 Spinorial derivatives

The expression of the spinorial derivatives of the basic superfields are obtained in general using the results of the Bianchi identities as well as the algebra of covariant derivatives and they are called also constituency equations. Let us sum up them here.

Spinorial derivatives of the scalars:

$$\begin{aligned}
\mathcal{D}_\varphi^{\mathrm{F}} T^{[\mathrm{DC}]\mathbf{u}} &= -\frac{\eta}{2(3!)}\varepsilon^{\mathrm{FDCIJKLM}} T_{[\mathrm{KLM}]\varphi} T_{[\mathrm{IJ}]}{}^{\mathbf{u}} && \text{(A.26)}\\
\mathcal{D}_{\mathrm{F}}^{\dot\varphi} T^{[\mathrm{DC}]\mathbf{u}} &= -\frac{1}{2(3!)}\delta^{\mathrm{DCI\,J}}_{\mathrm{F\,KLM}} T^{[\mathrm{KLM}]\dot\varphi} T_{[\mathrm{IJ}]}{}^{\mathbf{u}} && \text{(A.27)}\\
\mathcal{D}_\varphi^{\mathrm{F}} T_{[\mathrm{DC}]}{}^{\mathbf{u}} &= -\frac{1}{2(3!)}\delta^{\mathrm{F\,KLM}}_{\mathrm{DCI\,J}} T_{[\mathrm{KLM}]\varphi} T^{[\mathrm{IJ}]\mathbf{u}} && \text{(A.28)}\\
\mathcal{D}_{\mathrm{F}}^{\dot\varphi} T_{[\mathrm{DC}]}{}^{\mathbf{u}} &= -\frac{\bar\eta}{2(3!)}\varepsilon_{\mathrm{FDCIJKLM}} T^{[\mathrm{KLM}]\dot\varphi} T_{[\mathrm{IJ}]\mathbf{u}}\,. && \text{(A.29)}
\end{aligned}$$

Notice, that the spinorial derivatives of the scalars T are contained by definition in the spinorial components of the one-form $\Omega = (DT)S$. Therefore, the above expressions are equivalent to the relations

$$\mathcal{D}_\varphi^{\mathrm{F}} T = \Omega_\varphi^{\mathrm{F}} T\,, \qquad \mathcal{D}_{\mathrm{F}}^{\dot\varphi} T = \Omega_{\mathrm{F}}^{\dot\varphi} T\,, \tag{A.30}$$

with $\Omega_\varphi^{\mathrm{F}}$ and $\Omega_{\mathrm{F}}^{\dot\varphi}$ given by the equations (A.10) and (A.11).

Spinorial derivatives of the gravigini superfields:

$$\begin{aligned}
\mathcal{D}_\varphi^{\mathrm{F}} T_{[\mathrm{CBA}]\alpha} &= -4\delta^{\mathrm{FI\,J}}_{\mathrm{CBA}}(\sigma^{ba}\epsilon)_{\varphi\alpha}\mathcal{F}^{+}_{ba[\mathrm{IJ}]} + \frac{\bar\eta}{4!}\epsilon_{\varphi\alpha}\varepsilon_{\mathrm{CBAIJKLM}}\left(T^{[\mathrm{FIJ}]}T^{[\mathrm{KLM}]}\right) && \text{(A.31)}\\
\mathcal{D}_{\mathrm{F}}^{\dot\varphi} T_{[\mathrm{CBA}]\alpha} &= -4\mathrm{i}(\bar\sigma^f\epsilon)^{\dot\varphi}{}_\alpha \Omega_{f[\mathrm{FCBA}]} && \text{(A.32)}\\
\mathcal{D}_\varphi^{\mathrm{F}} T^{[\mathrm{CBA}]\dot\alpha} &= -4\mathrm{i}(\sigma^f\epsilon)_\varphi{}^{\dot\alpha}\Omega_f{}^{[\mathrm{FCBA}]} && \text{(A.33)}\\
\mathcal{D}_{\mathrm{F}}^{\dot\varphi} T^{[\mathrm{CBA}]\dot\alpha} &= -4\delta^{\mathrm{CBA}}_{\mathrm{FI\,J}}(\bar\sigma^{ba}\epsilon)^{\dot\varphi\dot\alpha}\mathcal{F}^{-\,[\mathrm{IJ}]}_{ba} + \frac{\eta}{4!}\epsilon^{\dot\varphi\dot\alpha}\varepsilon^{\mathrm{CBAIJKLM}}\left(T_{[\mathrm{FIJ}]}T_{[\mathrm{KLM}]}\right). && \text{(A.34)}
\end{aligned}$$

Spinorial derivatives of Ω_a, the field strength of the scalars:

$$\begin{aligned}
\mathcal{D}_\varphi^{\mathrm{F}}\Omega_e^{[\mathrm{DCBA}]} &= -\frac{\eta}{12}\varepsilon^{\mathrm{FDCBAIJK}}\mathcal{D}_e T_{[\mathrm{IJK}]\varphi} + \frac{\mathrm{i}}{3}\delta^{\mathrm{DCBA}}_{\mathrm{EI\,J\,K}}\left(\sigma^f T^{[\mathrm{IJK}]}\right)_\varphi F_{fe}{}^{[\mathrm{FE}]} - \\
&\quad -\frac{\eta}{12}\varepsilon^{\mathrm{EDCBAIJK}}\left[T_{[\mathrm{IJK}]\varphi}T_e^{\mathrm{F}}{}_{\mathrm{E}} + 2\left(\sigma_{ef}T_{[\mathrm{IJK}]}\right)_\varphi U^{f\mathrm{F}}{}_{\mathrm{E}}\right] && \text{(A.35)}\\
\mathcal{D}_{\mathrm{F}}^{\dot\varphi}\Omega_e^{[\mathrm{DCBA}]} &= -\frac{1}{12}\delta^{\mathrm{DCBA}}_{\mathrm{FI\,J\,K}}\mathcal{D}_e T^{[\mathrm{IJK}]\dot\varphi} + \frac{\mathrm{i}\eta}{3}\varepsilon^{\mathrm{EDCBAIJK}}\left(\bar\sigma^f T_{[\mathrm{IJK}]}\right)^{\dot\varphi} F_{fe[\mathrm{FE}]} + \\
&\quad +\frac{1}{12}\delta^{\mathrm{DCBA}}_{\mathrm{EI\,J\,K}}\left[T^{[\mathrm{IJK}]\dot\varphi}T_e^{\mathrm{E}}{}_{\mathrm{F}} + 2\left(\bar\sigma_{ef}T^{[\mathrm{IJK}]}\right)^{\dot\varphi} U^{f\mathrm{E}}{}_{\mathrm{F}}\right] && \text{(A.36)}\\
\mathcal{D}_\varphi^{\mathrm{F}}\Omega_{e[\mathrm{DCBA}]} &= -\frac{1}{12}\delta^{\mathrm{FI\,J\,K}}_{\mathrm{DCBA}}\mathcal{D}_e T_{[\mathrm{IJK}]\varphi} + \frac{\mathrm{i}\eta}{3}\varepsilon_{\mathrm{EDCBAIJK}}\left(\sigma^f T^{[\mathrm{IJK}]}\right)_\varphi F_{fe}{}^{[\mathrm{FE}]} - \\
&\quad -\frac{1}{12}\delta^{\mathrm{EI\,J\,K}}_{\mathrm{DCBA}}\left[T_{[\mathrm{IJK}]\varphi}T_e^{\mathrm{F}}{}_{\mathrm{E}} + 2\left(\sigma_{ef}T_{[\mathrm{IJK}]}\right)_\varphi U^{f\mathrm{F}}{}_{\mathrm{E}}\right] && \text{(A.37)}\\
\mathcal{D}_{\mathrm{F}}^{\dot\varphi}\Omega_{e[\mathrm{DCBA}]} &= -\frac{\bar\eta}{12}\varepsilon_{\mathrm{FDCBAIJK}}\mathcal{D}_e T^{[\mathrm{IJK}]\dot\varphi} + \frac{\mathrm{i}}{3}\delta^{\mathrm{EI\,J\,K}}_{\mathrm{DCBA}}\left(\bar\sigma^f T_{[\mathrm{IJK}]}\right)^{\dot\varphi} F_{fe[\mathrm{FE}]} + \\
&\quad +\frac{\bar\eta}{12}\varepsilon_{\mathrm{EDCBAIJK}}\left[T^{[\mathrm{IJK}]\dot\varphi}T_e^{\mathrm{E}}{}_{\mathrm{F}} + 2\left(\bar\sigma_{ef}T^{[\mathrm{IJK}]}\right)^{\dot\varphi} U^{f\mathrm{E}}{}_{\mathrm{F}}\right]. && \text{(A.38)}
\end{aligned}$$

Spinorial derivatives of the self-dual and anti-self-dual super-covariant field strength:

$$\begin{aligned}
\mathcal{D}_\varphi^{\mathrm{F}} F_{ba}^{-[\mathrm{BA}]} &= \frac{\mathrm{i}}{8}(\sigma^f\bar\sigma_{ba})_{\varphi\dot\alpha}\times \\
&\times\left[\mathcal{D}_f T^{[\mathrm{FBA}]\dot\alpha} + 2T^{[\mathrm{GBA}]\dot\alpha}U_f{}^{\mathrm{F}}{}_{\mathrm{G}} + \oint^{\mathrm{FBA}} T^{[\mathrm{GBA}]\dot\alpha}(T-U)_f{}^{\mathrm{F}}{}_{\mathrm{G}}\right] \qquad (A.39)
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\dot\varphi}_{\mathrm{F}} F_{ba}^{-[\mathrm{BA}]} &= (\epsilon\bar\sigma_{ba})_{\dot\beta\dot\alpha}\delta^{[\mathrm{B}}_{\mathrm{F}}\Sigma^{(\dot\varphi\dot\beta\dot\alpha)\mathrm{A}]} - \frac{\mathrm{i}}{8}\left[\Omega_f{}^{[\mathrm{BAIJ}]}\delta^{\mathrm{K}}_{\mathrm{F}} + \frac{1}{9}\delta^{\mathrm{BA}}_{\mathrm{F}\,\mathrm{G}}\Omega_f{}^{[\mathrm{GIJK}]}\right]\left(\bar\sigma_{ba}\bar\sigma^f T_{[\mathrm{IJK}]}\right)^{\dot\varphi} + \\
&+\frac{\bar\eta}{3!(4!)^2}\varepsilon_{\mathrm{FEIJKLMN}}\times \\
&\times\left[\left(T^{[\mathrm{IJK}]}\bar\sigma_{ba}T^{[\mathrm{LMN}]}\right)T^{[\mathrm{EBA}]\dot\varphi} - 2\left(T^{[\mathrm{IJK}]}\bar\sigma_{ba}T^{[\mathrm{EBA}]}\right)T^{[\mathrm{LMN}]\dot\varphi}\right] + \qquad (A.40)\\
&+\frac{\bar\eta}{4(4!)^2}\left[\frac{1}{9}\delta^{\mathrm{BA}}_{\mathrm{FN}}\delta^{\mathrm{DC}}_{\mathrm{KL}} + \delta^{\mathrm{BA}}_{\mathrm{KL}}\delta^{\mathrm{DC}}_{\mathrm{FN}}\right]\varepsilon_{\mathrm{DCMIJF_1F_2F_3}}(T^{[\mathrm{NLM}]}T^{[\mathrm{F_1F_2F_3}]})(\bar\sigma_{ba}T^{[\mathrm{IJK}]})^{\dot\varphi}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\mathrm{F}}_{\varphi} F^{+}_{ba[\mathrm{BA}]} &= (\epsilon\sigma_{ba})^{\beta\alpha}\delta^{\mathrm{F}}_{[\mathrm{B}}\Sigma_{(\varphi\beta\alpha)\mathrm{A}]} - \frac{\mathrm{i}}{8}\left[\Omega_{f[\mathrm{BAIJ}]}\delta^{\mathrm{F}}_{\mathrm{K}} + \frac{1}{9}\delta^{\mathrm{FG}}_{\mathrm{BA}}\Omega_{f[\mathrm{GIJK}]}\right]\left(\sigma_{ba}\sigma^f T^{[\mathrm{IJK}]}\right)_{\varphi} + \\
&+\frac{\eta}{3!(4!)^2}\varepsilon^{\mathrm{FEIJKLMN}}\times \qquad (A.41)\\
&\times\left[\left(T_{[\mathrm{IJK}]}\sigma_{ba}T_{[\mathrm{LMN}]}\right)T_{[\mathrm{EBA}]\varphi} - 2\left(T_{[\mathrm{IJK}]}\sigma_{ba}T_{[\mathrm{EBA}]}\right)T_{[\mathrm{LMN}]\varphi}\right] + \\
&+\frac{\eta}{4(4!)^2}\left[\frac{1}{9}\delta^{\mathrm{FN}}_{\mathrm{BA}}\delta^{\mathrm{KL}}_{\mathrm{DC}} + \delta^{\mathrm{KL}}_{\mathrm{BA}}\delta^{\mathrm{FN}}_{\mathrm{DC}}\right]\varepsilon^{\mathrm{DCMIJF_1F_2F_3}}(T_{[\mathrm{NLM}]}T_{[\mathrm{F_1F_2F_3}]})(\sigma_{ba}T_{[\mathrm{IJK}]})_{\varphi}
\end{aligned}$$

$$\mathcal{D}^{\dot\varphi}_{\mathrm{F}} F^{+}_{ba[\mathrm{BA}]} = \frac{\mathrm{i}}{8}(\bar\sigma^f\sigma_{ba})^{\alpha\dot\varphi}\left[\mathcal{D}_f T_{[\mathrm{FBA}]\alpha} - 2T_{[\mathrm{GBA}]\alpha}U_f{}^{\mathrm{G}}{}_{\mathrm{F}} - \oint_{\mathrm{FBA}} T_{[\mathrm{GBA}]\alpha}(T-U)_f{}^{\mathrm{G}}{}_{\mathrm{F}}\right] \qquad (A.42)$$

References

[1] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1983.

[2] S.J.Gates, M.T. Grisaru, M. Roček, and W. Siegel, *Superspace*, Benjamin/Cummings, London 1983.

[3] B. de Wit and D.Z. Freedman, *On* SO(8) *extended supergravity*, *Nucl. Phys.* **B 130** (1977) 105.

[4] E. Cremmer and B. Julia, *The $N = 8$ supergravity theory, 1. The lagrangian*, *Phys. Lett.* **B 80** (1978) 48.

[5] E. Cremmer and B. Julia, *The* SO(8) *supergravity*, *Nucl. Phys.* **B 159** (1979) 141.

[6] B. de Wit and H. Nicolai, *$N = 8$ supergravity*, *Nucl. Phys.* **B 208** (1982) 323.

[7] L. Brink and P. Howe, *The $N = 8$ supergravity in superspace*, *Phys. Lett.* **B 80** (1979) 268.

[8] P.S. Howe and U. Lindström, *Higher order invariants in extended supergravity*, *Nucl. Phys.* **B 181** (1981) 487.

[9] W. Siegel, *On-shell O(N) supergravity in superspace*, *Nucl. Phys.* **B 177** (1981) 325.

[10] P.S. Howe, *Supergravity in superspace*, *Nucl. Phys.* **B 199** (1982) 309.

[11] A. Kiss and E. Loyer, *$N = 8$ supergravity in central charge superspace*, hep-th/0209035.

[12] G. Akemann, R. Grimm, M. Hasler and C. Herrmann, $N = 2$ *central charge superspace and a minimal supergravity multiplet*, *Class. and Quant. Grav.* **16** (1999) 1617 [hep-th/9812026].

[13] R. Grimm, C. Herrmann and A. Kiss, $N = 4$ *supergravity with antisymmetric tensor in central charge superspace*, *Class. and Quant. Grav.* **18** (2001) 1027 [hep-th/0009201].

[14] B. Belinicher, *Relativistic wave equations and lagrangian formalism for particles of arbitrary spin*, *Theor. Math. Phys.* **20** (1974) 849.

[15] S. Weinberg, *The quantum theory of fields, vol. 1: Foundations.* University Press, Cambridge 1995.

[16] S. Gurrieri and A. Kiss, *Equations of motion for* $N = 4$ *supergravity with antisymmetric tensor from its geometric description in central charge superspace*, *J. High Energy Phys.* **02** (2002) 040 [hep-th/0201234].

[17] J. Gates, S.J. and R. Grimm, *Consequences of conformally covariant constraints for* $N > 4$ *superspace*, *Phys. Lett.* **B 133** (1983) 192.

[18] L.-L. Chau and C.-S. Lim, *Geometrical constraints and equations of motion in extended supergravity*, *Phys. Rev. Lett.* **56** (1986) 294.